NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3890

ON POSSIBLE SIMILARITY SOLUTIONS FOR THREE-DIMENSIONAL INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS

III - SIMILARITY WITH RESPECT TO STATIONARY POLAR COORDINATES FOR SMALL ANGLE VARIATION

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SUMMARY

Approximate solutions are obtained describing mainstream flows confined to regions of small angle variation over flat surfaces for three-dimensional, laminar, incompressible, thin boundary-layer flows having similarity with respect to stationary polar coordinate systems. The solutions, summarized in a table, include accelerating or decelerating flows and stagnation-point, spiral, or circular flows. An experimental comparison of limiting overturning at the wall showed good agreement for the first 10° of turning of circular mainstream flow.

INTRODUCTION

In addition to providing an insight into secondary-flow behavior associated with laminar boundary-layer flows, the experimental investigations of references 1 to 3 demonstrate that the information thus obtained for laminar flows can be used to interpret and to correlate flow measurements taken in turbomachines at operational conditions. These experimental investigations thereby provide an important link between applied turbomachine research and the similarity-type boundary-layer analyses developed in references 4 to 12. The link is further strengthened by the combined theoretical and experimental investigation of reference 13. In reference 13, boundary-layer similarity solutions are obtained for main flows consisting of streamline translates (i.e., the entire streamline pattern can be obtained by translating any particular streamline parallel to the leading edge), and the theoretical predictions of boundary-layer overturning (more than mainstream turning) near the surface are in close agreement with experimental results obtained by tracing the boundary-layer streamlines with smoke flow-visualization techniques.

Using a generalized similarity variable η, reference 14 extends these results analytically to obtain all possible flows with boundary layers having classical similarity with respect to stationary rectangular coordinates. The dimensionless boundary-layer velocity components in the plane of the surface are assumed to have similarity with respect to their respective coordinates. This similarity is expressed by means of two suitably defined functions of the similarity variable. The boundary-layer equations are then transformed to equations involving the mainstream flow components, their derivatives, the similarity functions, and their derivatives. All the mainstream flows are then determined for which the transformed boundary-layer equations reduce to ordinary differential equations in the similarity functions and their derivatives. Four distinct families of such mainstream flows are obtained in reference 14, including cases of accelerating or decelerating flows for quite general streamline paths. The main-flow streamlines are not required to be translates, nor are they restricted to regions of small turning in reference 14.

In reference 15, solutions are obtained for the mainstream flows whose boundary-layer velocity components in the plane of the surface have similarity with respect to the corresponding polar coordinates. Thus, exact solutions are obtained for spiral, circular, and stagnation-point flow configurations with no restrictions on mainstream turning. For the solutions thus obtained, however, a proper leading edge cannot be defined. (A proper leading edge, which corresponds theoretically to a real physical leading edge, would be a line or curve of zero boundary-layer thickness on the surface, downstream of which the boundary layer develops.)

The present investigation extends the analysis of reference 15 by considering the flows in a sector-region of small central angle θ. The purpose of this investigation is to determine mainstream flow solutions for which the transformed boundary-layer equations reduce to ordinary differential equations. Solutions are obtained for four new families of mainstream flows with boundary layers having similarity with respect to the polar coordinates in the plane of the surface. Included here are cases for flows over well-defined leading edges. It is important to note that experimental investigations (refs. 1 to 3 and 13) indicate that in typical turbomachine configurations a large portion of the end-wall boundary layer at the inlet to a passage has completely crossed from the pressure to the suction side of the passage when the mainstream has been turned less than 30°. Thus, in describing physical flow, it appears not unreasonable to restrict the analysis to small central-angle sectors. The regions where this assumption might be considered reasonable are established by a theoretical and experimental comparison of the boundary-layer limiting flow deflection (ref. 13) in a circular two-dimensional channel.

SYMBOLS

a,b,c,C	constants
$F,F(\eta)$	function of similarity parameter, $u \equiv UF'(\eta)$
$\mathcal{F},\mathcal{F}(-\eta)$	function of similarity parameter, eq. (45)
$f(r),f^*(r)$	arbitrary functions of r
$\mathit{f}(r,\theta)$	arbitrary function of r and θ
$G,G(\eta)$	function of similarity parameter, $w \equiv WG'(\eta)$ for $W \neq 0$, $w \equiv \overline{W}G'(\eta)$ for $W = 0$
$g,g(r,\theta)$	function of coordinates r and θ
$h(\theta)$	function of θ
k,m,n	constants
r,θ,y	cylindrical coordinates
t	constant
U,W	mainstream velocity components in θ and r directions, respectively
u,v,w	boundary-layer velocity components in θ, y, r directions, respectively
$\overline{W},\overline{W}(r,\theta)$	function of coordinates r and θ, $w \equiv \overline{W}G'(\eta)$ for $W = 0$
γ	boundary-layer deflection at surface
η	similarity variable $\eta = yg(r,\theta)/\sqrt{\nu}$
ν	coefficient of kinematic viscosity

Subscripts:

$i = 1,2,3, \ldots$	index numbers

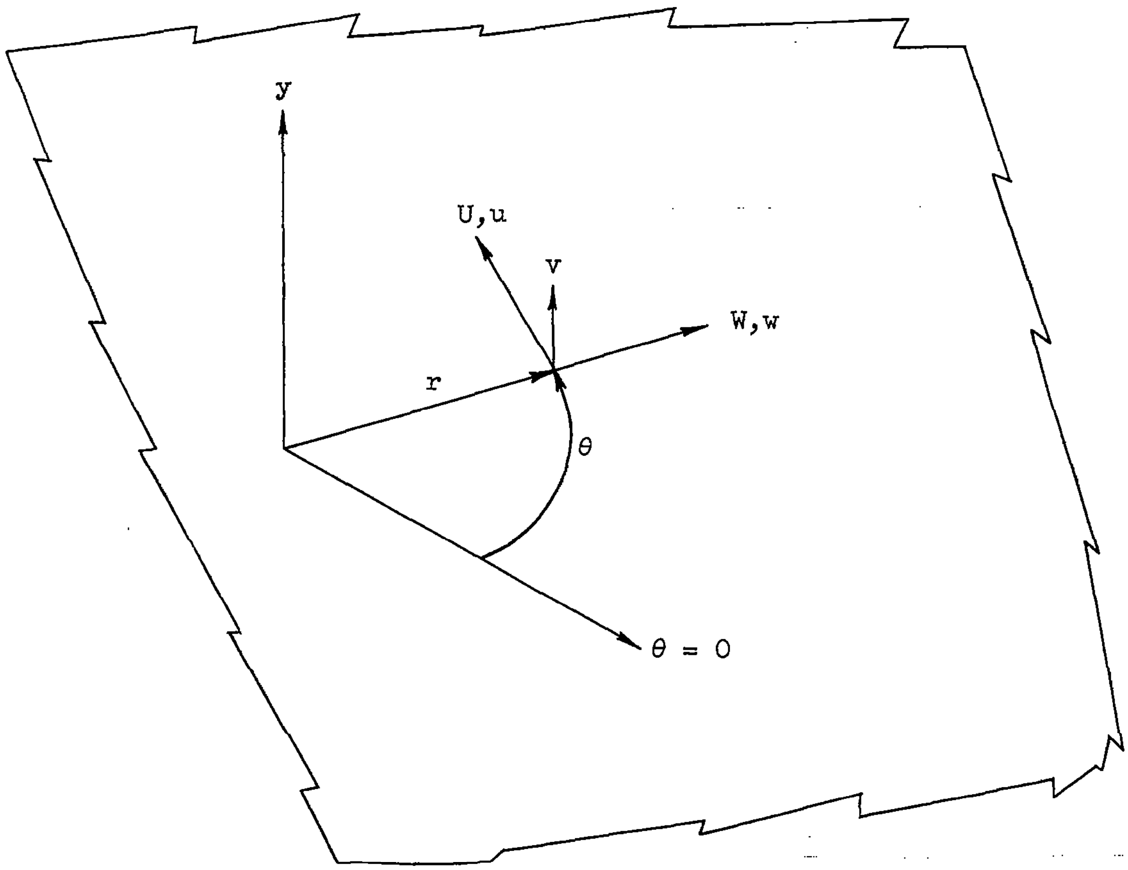
Superscripts:

Primes denote differentiation

ANALYSIS

Boundary-Layer Equations in Stationary Cylindrical Coordinates

The three-dimensional laminar incompressible thin boundary-layer equations in cylindrical coordinate form for flows over flat (or nearly flat) surfaces with stationary coordinate axes as shown here



are given by

$$\frac{uw}{r} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial y} - \nu\frac{\partial^2 u}{\partial y^2} = \frac{UW}{r} + \frac{U}{r}\frac{\partial U}{\partial \theta} + W\frac{\partial U}{\partial r} \tag{1a}$$

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in the tangential direction and

$$-\frac{u^2}{r}+\frac{u}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial r}+v\frac{\partial w}{\partial y}-\nu\frac{\partial^2 w}{\partial y^2}=-\frac{U^2}{r}+\frac{U}{r}\frac{\partial W}{\partial \theta}+W\frac{\partial W}{\partial r} \qquad (1b)$$

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in the radial direction, where u, w, and v are the boundary-layer velocity components in the θ, r, and y directions, respectively. Consistent with the restriction to thin boundary-layer flows over flat (or nearly flat) surfaces as required for the formulation of the boundary-layer equations (eq. (1)), the mainstream velocity components are

$$U = U(r,\theta) \qquad (2a)$$

$$W = W(r,\theta) \qquad (2b)$$

The equation of continuity for the boundary-layer flow is

$$\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial w}{\partial r}+\frac{w}{r}+\frac{\partial v}{\partial y}=0 \qquad (1c)$$

The appropriate boundary conditions are

$$u = w = v = 0 \quad \text{for} \quad y = 0 \qquad (1d)$$

$$\left.\begin{aligned} u &\rightarrow U \\ w &\rightarrow W \end{aligned}\right\} \text{as} \quad y \rightarrow \infty \qquad (1e)$$

Similarity with Respect to Stationary Polar Coordinates

The boundary-layer equations may be transformed by the use of a generalized space variable

$$\eta \equiv \frac{y}{\sqrt{\nu}}\, g(r,\theta) \qquad (3)$$

and by defining

$$u \equiv UF'(\eta) \qquad (4a)$$

$$w \equiv WG'(\eta) \qquad (4b)$$

to a new system of coordinates r, θ, and η. The definitions (4a) and (4b) are the requirements for similarity of the boundary-layer velocity components in the plane of the surface with respect to their corresponding polar coordinates.

The choice of polar coordinates (as in ref. 15) requires an additional precaution beyond those needed for similarity with respect to rectangular coordinates. In rectangular coordinates when either U or W is zero (see ref. 14), the mainstream flows are straight, there is no secondary-flow overturning in the boundary layer, and complete similarity solutions have been obtained for the equations of the resulting two-dimensional boundary-layer flows (refs. 4 and 5). In the present case, however, when the mainstream radial component $W = 0$, there is curvature of the mainstream flow, $U \neq 0$, and three-dimensional boundary-layer overturning results (i.e., $w \neq 0$, for $W = 0$). Under these conditions, equation (4b) does not apply. Instead, a new function $\overline{W}(r,\theta)$ is defined for 4234

$$w \equiv \overline{W}G'(\eta), \ \overline{W} \neq 0 \tag{4c}$$

Accordingly, it is convenient to treat $W \neq 0$ flows separately from $W = 0$, $\overline{W} \neq 0$ flows.

<u>$W \neq 0$.</u> - When the mainstream flow has both U and W components, the corresponding boundary-layer velocity components are defined by equations (4a) and (4b) as functions of the similarity parameter η. Corresponding to the conditions of no flow at the surface (eq. (1d)), the boundary conditions on F' and G' are

$$F'(0) = G'(0) = 0 \tag{5a}$$

Corresponding to the condition (1e) that the u and w boundary-layer components merge smoothly into main-flow components U and W, respectively,

$$\left.\begin{aligned} \lim_{\eta \to \infty} F'(\eta) &= 1 \\ \lim_{\eta \to \infty} G'(\eta) &= 1 \end{aligned}\right\} \tag{5b}$$

Now v may be determined by integration of the continuity equation using (4a) and (4b):

$$v = \frac{-\sqrt{\nu}}{g}\left[\left(\frac{1}{r}\frac{\partial U}{\partial \theta} - \frac{U}{r}\frac{\partial \ln g}{\partial \theta}\right)F + \left(\frac{\partial W}{\partial r} - W\frac{\partial \ln g}{\partial r} + \frac{W}{r}\right)G\right] - \frac{U}{r}\frac{\partial \ln g}{\partial \theta}yF' - W\frac{\partial \ln g}{\partial r}yG' + f(r,\theta) \tag{6}$$

where $f(r,\theta)$ is an arbitrary function arising from integration.

In order that $v = 0$ for $y = 0$ as required, it is possible without loss of generality to set the boundary conditions

$$F(0) = G(0) = 0, \text{ and } f(r,\theta) = 0 \tag{5c}$$

(See appendix C, ref. 14, for a discussion of the necessary and sufficient boundary conditions.)

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Upon substitution of equations (4a), (4b), and (6), equation (1a) becomes (for $W \neq 0$)

$$\frac{W}{r}\left(F'G' - GF'' - 1\right) + \frac{1}{r}\frac{\partial U}{\partial \theta}\left(F'^2 - FF'' - 1\right) + W\frac{\partial \ln U}{\partial r}\left(F'G' - 1\right) +$$

$$\frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta}FF'' + \frac{W}{2}\frac{\partial \ln g^2}{\partial r}GF'' - \frac{\partial W}{\partial r}GF'' - g^2F''' = 0 \tag{7}$$

and equation (1b) becomes

$$\frac{U^2}{Wr}\left(1 - F'^2\right) + \frac{U}{r}\frac{\partial \ln W}{\partial \theta}\left(F'G' - 1\right) + \frac{\partial W}{\partial r}\left(G'^2 - GG'' - 1\right) - \frac{1}{r}\frac{\partial U}{\partial \theta}FG'' +$$

$$\frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta}FG'' + \frac{W}{2}\frac{\partial \ln g^2}{\partial r}GG'' - \frac{W}{r}GG'' - g^2G''' = 0 \tag{8}$$

As in references 14 and 15, the purpose of this investigation is to determine mainstream flow solutions for which the transformed equations (7) and (8) reduce to ordinary differential equations. As an extension of reference 15, the present analysis considers flows restricted to a sector-region of small central angle θ. The mainstream flow conditions are sought which make the coefficients of the functions of η proportional.

The most general approach would be to rewrite (7) and (8), grouping the coefficients of like terms in G,F and their derivatives, and then to require proportionality of these grouped coefficients. It can be shown, however, that no cases arise beyond those obtained more simply by requiring proportionality of the individual coefficients in (7) and (8). Under these ordinary differential equation conditions (abbreviated to o.d.e. conditions), the common variable terms in the equations may be divided out, leaving ordinary differential equations for F and G. The actual numerical solutions of the ordinary differential equations are not attempted herein.

For convenience, the coefficients for the functions of η in equations (7) and (8) are presented here in the order of their appearance. With $W \neq 0$, they are

1 $\frac{W}{r}$ 6 $\frac{\partial W}{\partial r}$

2 $\frac{1}{r}\frac{\partial U}{\partial \theta}$ 7 g^2

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3 $W\frac{\partial \ln U}{\partial r}$ 8 $\frac{U^2}{Wr}$

4 $\frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta}$ 9 $\frac{U}{r}\frac{\partial \ln W}{\partial \theta}$

5 $\frac{W}{2}\frac{\partial \ln g^2}{\partial r}$

The o.d.e. conditions require these nine coefficients to be proportional to each other.

$W = 0$. - When $W = 0$, the corresponding boundary-layer equations (1a) and (1b) become

$$\frac{uw}{r} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial y} - \nu\frac{\partial^2 u}{\partial y^2} = \frac{U}{r}\frac{\partial U}{\partial \theta} \tag{9a}$$

$$-\frac{u^2}{r} + \frac{u}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial r} + v\frac{\partial w}{\partial y} - \nu\frac{\partial^2 w}{\partial y^2} = -\frac{U^2}{r} \tag{9b}$$

The equation of continuity for the boundary-layer flow remains unchanged:

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial r} + \frac{w}{r} + \frac{\partial v}{\partial y} = 0 \tag{1c}$$

The boundary conditions now are

$$u = w = v = 0 \quad \text{for} \quad y = 0$$

$$\left.\begin{aligned} u &\to U \\ w &\to 0 \end{aligned}\right\} \text{ as } y \to \infty \tag{9c}$$

For main flows such that $W = 0$, define

$$u \equiv UF'(\eta) \tag{4a}$$

$$w \equiv \overline{W}G'(\eta) \tag{4c}$$

where $\overline{W} = \overline{W}(r,\theta) \neq 0$. The boundary conditions on F' and G' required to satisfy boundary conditions on u and w in equations (9) ($W = 0$, $\overline{W}(r,\theta) \neq 0$) are

$$F'(0) = G'(0) = 0 \tag{10a}$$

$$\left.\begin{aligned} \lim_{\eta \to \infty} F'(\eta) &= 1 \\ \lim_{\eta \to \infty} G'(\eta) &= 0 \end{aligned}\right\} \tag{10b}$$

The expression for v obtained by integration of the continuity equation (1c) is the same as (6), with W being replaced by $\overline{W}$:

$$v = \frac{-\sqrt{\nu}}{g}\left[\left(\frac{1}{r}\frac{\partial U}{\partial \theta} - \frac{U}{r}\frac{\partial \ln g}{\partial \theta}\right)F + \left(\frac{\partial \overline{W}}{\partial r} - \overline{W}\frac{\partial \ln g}{\partial r} + \frac{\overline{W}}{r}\right)G\right] - \frac{U}{r}\frac{\partial \ln g}{\partial \theta}yF' - \overline{W}\frac{\partial \ln g}{\partial r}yG' + f(r,\theta) \tag{11}$$

As before, the boundary conditions chosen as sufficient to provide that $v = 0$ for $y = 0$ are

$$\left.\begin{aligned} F(0) = G(0) &= 0 \\ f(r,\theta) &= 0 \end{aligned}\right\} \tag{10c}$$

Substitution of equations (4) and (11) into (9a) and (9b) produces

$$\frac{\overline{W}}{r}(F'G' - GF'') + \frac{1}{r}\frac{\partial U}{\partial \theta}(F'^2 - FF'' - 1) + \overline{W}\frac{\partial \ln U}{\partial r}F'G' + \frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta}FF'' + \frac{\overline{W}}{2}\frac{\partial \ln g^2}{\partial r}GF'' - \frac{\partial \overline{W}}{\partial r}GF'' - g^2F''' = 0 \tag{12}$$

$$\frac{U^2}{\overline{W}r}(1 - F'^2) + \frac{U}{r}\frac{\partial \ln \overline{W}}{\partial \theta}F'G' + \frac{\partial \overline{W}}{\partial r}(G'^2 - GG'') - \frac{1}{r}\frac{\partial U}{\partial \theta}FG'' + \frac{U}{2r}\frac{\partial \ln g^2}{\partial \theta}FG'' + \frac{\overline{W}}{2}\frac{\partial \ln g^2}{\partial r}GG'' - \frac{\overline{W}}{r}GG'' - g^2G''' = 0 \tag{13}$$

The argument concerning determination of the o.d.e. conditions by means of relations between the coefficients of the functions of η in (12) and (13) remains unchanged. These coefficients, it may be noted, are the same as the coefficients for equations (7) and (8), respectively, with W replaced by $\overline{W}$.

Solutions for Small Variation of θ

As the flow is considered restricted to regions of small variation of θ (i.e., regions that are narrow sectors having small central angles), coefficients will be neglected that are of second and higher order in θ relative to the other coefficients. In order to do this, it is assumed that $U(r,\theta)$, $W(r,\theta)$, and $\overline{W}(r,\theta)$ are expressible as

$$U(r,\theta) = f_1^*(r)h_1(\theta)$$

$$W(r,\theta) = f_2^*(r)h_2(\theta), \quad W \neq 0$$

$$\overline{W}(r,\theta) = f_2^*(r)h_2(\theta), \quad W = 0$$

and that, in the region of interest chosen for convenience about $\theta \approx 0$, $h_1(\theta)$ and $h_2(\theta)$ are adequately represented by

$$h_1(\theta) \approx a\theta^t$$

$$h_2(\theta) \approx b\theta^m$$

Hence, U, W, and $\overline{W}$ in the neighborhood of $\theta = 0$ are considered to be defined by

$$\left.\begin{aligned} U(r,\theta) &= f_1(r)\theta^t \\ W(r,\theta) &= f_2(r)\theta^m \\ \overline{W}(r,\theta) &= f_2(r)\theta^m \end{aligned}\right\} \tag{14}$$

As will be seen later, in some cases this assumption further results in solutions that have properly defined leading edges. Solutions are obtained for the main flows in the following manner.

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By substitution, the coefficients of the functions of F and G in equations (7) and (8) for $W \neq 0$ or in equations (12) and (13) for $W = 0$ become

$$\left.\begin{array}{ll}
① & \dfrac{f_2(r)}{r}\,\theta^m \\[2ex]
② & \dfrac{t}{r}\,f_1(r)\theta^{t-1} \\[2ex]
③ & \dfrac{f_2(r)}{f_1(r)}\,f_1'(r)\theta^m \\[2ex]
④ & \dfrac{f_1(r)}{2r}\,\dfrac{\partial \ln g^2}{\partial \theta}\,\theta^t \\[2ex]
⑤ & \dfrac{f_2(r)}{2}\,\dfrac{\partial \ln g^2}{\partial r}\,\theta^m \\[2ex]
⑥ & f_2'(r)\theta^m \\[2ex]
⑦ & g^2 \\[2ex]
⑧ & \dfrac{(f_1(r))^2}{rf_2(r)}\,\theta^{2t-m} \\[2ex]
⑨ & m\,\dfrac{f_1(r)}{r}\,\theta^{t-1}
\end{array}\right\} \qquad (15)$$

As before (refs. 14 and 15), the objective is to find conditions (o.d.e. conditions) that make these coefficients proportional to one another. Then the common r and θ factors may be divided out, and the transformed equations reduce to ordinary differential equations. Here, however, with θ small, it is assumed that terms of second and higher order in θ relative to the rest are negligible.

The procedure will be to establish the relations between constants m and t (the powers of θ involved) or the assumptions concerning $f_1(r)$ or $f_2(r)$ that will lead to proportionality among the terms. The following possibilities are suggested.

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m = t - 1. - When $m = t - 1$, θ^m is a common factor of all the coefficients listed in (15) except ⑦ and may be divided out, yielding

$$\left.\begin{aligned}
&① \; \frac{f_2}{r} \\
&② \; \frac{m+1}{r} f_1 \\
&③ \; \frac{f_2 f_1'}{f_1} \\
&④ \; \frac{f_1}{2r} \frac{\partial \ln g^2}{\partial \theta} \theta \\
&⑤ \; \frac{f_2}{2} \frac{\partial \ln g^2}{\partial r} \\
&⑥ \; f_2' \\
&⑦ \; g^2 \theta^{-m} \\
&⑧ \; \frac{f_1^2}{r f_2} \theta^2 \\
&⑨ \; m \frac{f_1}{r}
\end{aligned}\right\} \tag{16}$$

Coefficient ⑦ is the coefficient of the F''' and G''' terms in the transformed boundary-layer equations. It must not be permitted to vanish, for that would reduce the order of the transformed equations, and the number of boundary conditions on F and G would then exceed the order of the equations. Consequently, ⑦ is made proportional to ⑥, with the result that

$$g^2 = a_1 f_2' \theta^m \tag{17}$$

Coefficient ④ is therefore independent of θ. Coefficient ⑧ is of second order in θ relative to the rest and is therefore neglected. From o.d.e. conditions on ① and ⑨,

$$f_1 = a_2 f_2 \tag{18}$$

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and, from ① and ⑥,

$$f_2 = a_3 r^{a_4} \tag{19}$$

Therefore, combining (14), (17), (18), and (19) and redefining the constants a_i for convenience results in

$$\left.\begin{aligned} U &= ar^n\theta^{m+1} \\ W &= br^n\theta^m, \; W \neq 0 \\ g^2 &= cr^{n-1}\theta^m \end{aligned}\right\} \tag{20}$$

The corresponding ordinary differential equations are obtained by substitution in equations (7) and (8):

$$b(n+1)(F'G' - 1) - \frac{b(n+3)}{2} GF'' + a(m+1)\left[(F')^2 - 1\right] - \frac{a(m+2)}{2} FF'' - cF''' = 0 \tag{21}$$

$$am(F'G' - 1) + bn\left[(G')^2 - 1\right] - \frac{b(n+3)}{2} GG'' - \frac{a(m+2)}{2} FG'' - cG''' = 0 \tag{22}$$

with equation (5) giving the boundary conditions. When $W = 0$, $\overline{W} = br^n\theta^m \neq 0$; and substitution into equations (12) and (13) yields

$$b(n+1) F'G' - \frac{b(n+3)}{2} GF'' + a(m+1)\left[(F')^2 - 1\right] - \frac{a(m+2)}{2} FF'' - cF''' = 0 \tag{23}$$

$$am(F'G') + bn(G')^2 - \frac{b(n+3)}{2} GG'' - \frac{a(m+2)}{2} FG'' - cG''' = 0 \tag{24}$$

with equation (10) giving the boundary conditions. It is important to note that the boundary conditions on G' are different for $W \neq 0$ in equations (21) and (22) from those for $W = 0$, $\overline{W} \neq 0$ in equations (23) and (24). While a more general case appears to be

$$m = t - k, \; k = 1,2,3$$

only the case discussed here, $k = 1$, has any practical significance—for this analysis. When $k \geq 2$, then, for small θ, U is negligible when compared with W, and the resultant flows are cases of $U = 0$.

m = t + 1. - When $m = t + 1$, substituting into (15) and dividing all terms by θ^{m-2} give

$$\left.\begin{array}{ll}
① & \dfrac{f_2}{r}\,\theta^2 \\[2ex]
② & \dfrac{m - 1}{r}\,f_1 \\[2ex]
③ & \dfrac{f_2}{f_1}\,f_1'\theta^2 \\[2ex]
④ & \dfrac{f_1}{2r}\,\dfrac{\partial \ln g^2}{\partial \theta}\,\theta \\[2ex]
⑤ & \dfrac{f_2}{2}\,\dfrac{\partial \ln g^2}{\partial r}\,\theta^2 \\[2ex]
⑥ & f_2'\theta^2 \\[2ex]
⑦ & g^2\theta^{2-m} \\[2ex]
⑧ & \dfrac{(f_1)^2}{rf_2} \\[2ex]
⑨ & \dfrac{m}{r}\,f_1
\end{array}\right\} \tag{25}$$

From the o.d.e. conditions on ⑦ and ⑧ it can be seen than g^2 has the form

$$g^2 = \frac{b_1}{r}\,f_1\theta^{m-2} \tag{26}$$

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Coefficient ④ is therefore independent of θ and so ①, ③, ⑤, and ⑥ are considered negligibly small. The modified coefficients are now written

$$\left.\begin{array}{ll} ① & 0 \\ ② & \frac{m-1}{r} f_1 \\ ③ & 0 \\ ④ & \frac{m-2}{2r} f_1 \\ ⑤ & 0 \\ ⑥ & 0 \\ ⑦ & \frac{b_1}{r} f_1 \\ ⑧ & \frac{(f_1)^2}{r f_2} \\ ⑨ & \frac{m}{r} f_1 \end{array}\right\} \tag{27}$$

From the o.d.e. conditions ⑧ and ⑨ it can be seen that

$$f_1 = b_2 f_2 \tag{28}$$

and no further restrictions on the form of f_1 or f_2 are required, as all coefficients are proportional. Then using (14), (26), and (28), U, W, and g^2 are written

$$\left.\begin{array}{l} U = af(r)\theta^{m-1} \\ W = bf(r)\theta^{m}, \; W \neq 0 \\ g^2 = \frac{c}{r} f(r)\theta^{m-2} \end{array}\right\} \tag{29}$$

and equations (7) and (8) become

$$a(m - 1) \left[(F')^2 - 1\right] - \frac{am}{2} FF'' - cF''' = 0 \tag{30}$$

$$-\frac{a^2}{b}\left[(F')^2 - 1\right] + am(F'G' - 1) - \frac{am}{2} FG'' - cG''' = 0 \tag{31}$$

The boundary conditions are given by equation (5).

When $W = 0$, $\overline{W} = bf(r)\theta^m \neq 0$, substitution into equations (12) and (13) produces

$$a(m - 1)\left[(F')^2 - 1\right] - \frac{am}{2} FF'' - cF''' = 0 \tag{32}$$

$$-\frac{a^2}{b}\left[(F')^2 - 1\right] + am(F'G') - \frac{am}{2} FG'' - cG''' = 0 \tag{33}$$

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Again, while

$$m = t + k,\ k = 1,2,3$$

appears more general, only $k = 1$ actually applies here. When $k \geqslant 2$, $W = 0$ under the present assumptions, and the resulting flow is equivalent to taking $b = 0$ in equations (20) or (29).

<u>$m = 0$</u>. - When $m = 0$, from o.d.e. conditions 1 and 8 and the assumption that terms of second or higher order in θ relative to other terms may be neglected, $t = 1, -1,$ or 0. The case $t = 1$ corresponds to the case of equation (20) and $t = -1$ to the case of equation (29). If $t = 0$, $U = U(r)$, f_1 is proportional to f_2, and therefore

$$\left.\begin{aligned} U &= ar^n \\ W \text{ (or } \overline{W}) &= br^n \\ g^2 &= cr^{n-1} \end{aligned}\right\} \tag{34}$$

This case and the resulting ordinary differential equations are the same as were obtained by taking $U = U(r)$, $m = 0$ in the exact solutions (ref. 15).

<u>$t = 0$</u>. - When $t = 0$, examination of the o.d.e. conditions discloses no cases not already obtained by the analysis.

<u>$f_1(r) = a$ or $f_2(r) = b$</u>. - Where $f_1(r)$ is taken to be a constant, a, $U = U(\theta)$ alone and the coefficients in (15) become

$$\left.\begin{array}{ll}
① & \dfrac{f_2}{r}\,\theta^m \\[2ex]
② & \dfrac{t}{r}\,\theta^{t-1} \\[2ex]
③ & 0 \\[2ex]
④ & \dfrac{1}{2r}\,\dfrac{\partial \ln g^2}{\partial \theta}\,\theta^t \\[2ex]
⑤ & \dfrac{f_2}{2}\,\dfrac{\partial \ln g^2}{\partial r}\,\theta^m \\[2ex]
⑥ & f_2'\theta^m \\[2ex]
⑦ & g^2 \\[2ex]
⑧ & \dfrac{1}{rf_2}\,\theta^{2t-m} \\[2ex]
⑨ & \dfrac{m}{r}\,\theta^{t-1}
\end{array}\right\} \tag{35}$$

Just as before, two significant possibilities occur, $m = t - 1$ and $m = t + 1$. When $m = t - 1$, equation (18) applies and f_2 is a constant. This corresponds, of course, to $n = 0$ (eq. (20)). When $m = t + 1$, the analysis leading to equation (28) applies, and the resultant flows correspond to $f(r)$ constant in equation (29).

When $f_2(r)$ is constant, the same results are obtained and $f_1(r)$ must likewise be constant.

RESULTS AND DISCUSSION

The analysis of three-dimensional, laminar, incompressible boundary-layer flows having similarity with respect to polar coordinates has led to two different categories of solutions. In the first are the solutions for mainstream flows described in reference 15. In the second category are perturbation-type solutions obtained here whose validity is restricted to regions of small variation of angle θ. The mainstream flows for the latter category are described by equations (20), (29), and (34).

The flows represented by equation (34) are actually a special case of the exact solutions of reference 15 and will not be discussed further. As a result of these analyses, table I has been prepared, which summarizes the four new cases obtained here of mainstream flows over a flat or slightly curved surface for which the boundary-layer flows have similarity with respect to polar coordinates.

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The cases are as follows:

Case I: $U = ar^n\theta^{m+1}$

$W = br^n\theta^m, \ b \neq 0$

Case II: $U = ar^n\theta^{m+1}$

$W = 0$

$\overline{W} = br^n\theta^m, \ b \neq 0$

Case III: $U = af(r)\theta^{m-1}$

$W = bf(r)\theta^m, \ b \neq 0$

Case IV: $U = af(r)\theta^{m-1}$

$W = 0$

$\overline{W} = bf(r)\theta^m, \ b \neq 0$

As described earlier, secondary flows exist even though the radial component of mainstream flow vanishes $(W = 0)$. For such cases, a function $\overline{W} = \overline{W}(r,\theta) \neq 0$ is defined, and the boundary-layer radial component of flow is expressed as

$$w = \overline{W}G'(\eta) \tag{4c}$$

The Mainstream

When $W \neq 0$, the mainstreams are spiral flows. For $W = 0$, circular mainstream flows are obtained.

U, W, $\overline{W}$. - In regions where the thin-boundary-layer theory is applicable, the mainstream is very nearly parallel to the surface; U and W are functions of r and θ only.

The analysis for the exact solutions for boundary-layer flows having similarity with respect to stationary polar coordinates (ref. 15) showed that only one form of U and W (or $\overline{W}$) is possible, that is

$$U = ar^n e^{m\theta}$$

$$W \text{ (or } \overline{W}) = br^n e^{m\theta} \tag{36}$$

The present analysis results in obtaining a much wider variety of flow solutions, represented by equations (20) and (29). In particular, there are no restrictions whatsoever on the functional relation of U and W (or $\overline{W}$) to r in cases III and IV (eq. (29)). An example might be chosen of circular main-flow streamlines where the inlet velocity U at the leading edge $\theta = 0$ varies in a sinusoidal fashion. The secondary-flow overturning would then cause the boundary layer to pass through regions where the main-flow velocity is alternately increasing and decreasing. As will be seen later in the discussion of the boundary-layer thickness and $g(r,\theta)$, the boundary layer will correspondingly become thinner or thicker.

The small-angle solutions were obtained by assuming that second and higher power terms of θ could be neglected. In all these cases, U and W (or $\overline{W}$) are obtained as products of powers of r and θ. In cases I and II W/U (or $\overline{W}$/U) is proportional to $1/\theta$. In cases III and IV, W/U (or $\overline{W}$/U) is proportional to θ.

Projection of main-flow streamline on surface. - The equation for the projection of the main-flow streamline on the surface for $W \neq 0$ may be obtained by integrating

$$\frac{W}{U} = \frac{dr}{r\, d\theta} \tag{37}$$

Whenever $W \neq 0$ and $U \neq 0$, spiral mainstream flows result (cases I and III). For $W = 0$, circular main-flow streamlines result (cases II and IV).

Slope of projected streamlines. - The slope of the projected streamline with respect to $\theta = 0$ (the tangent of the angle between the tangent to the projected streamline curve at a point and the line $\theta = 0$) may be obtained from

$$\text{slope} = \frac{\frac{dr}{d\theta} \tan\theta + r}{\frac{dr}{d\theta} - r \tan\theta} \tag{38}$$

It is found by substitution into equation (38) that the slope is independent of radial position r for all cases.

Irrotationality. - For mainstream flows considered here and in regions of thin boundary layers, as required for this analysis, only the component of vorticity normal to the surface

$$\frac{1}{r}\frac{\partial W}{\partial \theta} - \frac{\partial U}{\partial r} - \frac{U}{r} \tag{39a}$$

can be much different from zero (ref. 14). The values of the constants specified under the listing "Irrotationality" (table I) were obtained in each case from

$$\frac{1}{r}\frac{\partial W}{\partial \theta} - \frac{\partial U}{\partial r} - \frac{U}{r} = 0 \tag{39b}$$

These values set the conditions for nearly irrotational mainstream flows.

The Boundary Layer

As discussed in references 14 and 15, the physical interpretation of the boundary-layer behavior that the mathematical representations describe is best found by examining the behavior of η and in particular $g(r,\theta)$.

The boundary-layer thickness on the surface at a point r,θ is inversely proportional to $g(r,\theta)$ at the point. In order for the theoretical boundary layer to have a beginning at a leading edge with zero thickness, as in a real fluid, there should be a line along the surface for which $g(r,\theta)$ is infinite while the velocities remain finite. In the exact solutions presented in reference 15, this occurs in the finite part of the plane only at the point $r = 0$ for values of $n < 1$. For $n > 1$ the boundary layer in reference 15 may be considered to have a "beginning" only at $r = \infty$. However, the mainstream velocities there take on "infinite" values.

In the present report (flows for small angle θ), for cases I and II when $-1 \leq m < 0$, $g(r,\theta)$ and hence η becomes infinite along the line $\theta = 0$, while U is finite as required. In case I, however, W (and w) are unbounded, so a proper leading edge does not exist there. In case II, $W = 0$ but $\overline{W}$ is unbounded at $\theta = 0$. In cases III and IV, for $1 \leq m < 2$, there is a properly defined leading edge at $\theta = 0$, where $g(r,\theta)$ takes on infinite values and where U, u, W, and w remain finite. For $1 < m < 2$, in cases III and IV, $U = W = 0$ at the leading edge. For $m = 1$, U is independent of θ and may be different from zero at the leading edge.

In cases I and II for $m > 0$, and in cases III and IV for $m > 2$, along the line $\theta = 0$, the mainstream and the boundary-layer velocity components are all zero. Even though the boundary-layer velocities match the mainstream in these situations, a proper leading edge does not exist there because $g(r,\theta)$ equals zero, corresponding to an "infinitely" thick boundary layer. As in reference 14, such accelerated-flow cases may be considered appropriately by confining the discussion to regions where the requirement of thin boundary layers is satisfied.

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The Ordinary Differential Equations

The actual numerical solutions of the ordinary differential equations are beyond the scope of the present investigation. The literature contains examples of numerical solutions for particular values of the constants. Some of these examples are noted in the listing "Comments and References" associated with each case in table I.

The present analysis simply derives the ordinary differential equations that can be obtained with the underlying assumptions. In any particular case of interest for which the equations are appropriate, the existence of the numerical solution and its computation must be obtained individually. Nevertheless, some general remarks (in part repeating material from ref. 14 here for convenience) can be made here (as in ref. 11) concerning the numerical solutions.

Separation of F and G. - Under certain choices of the free constants involved, the functions F and G are separable; that is, one equation of the pair of ordinary differential equations will contain terms in only one of these functions and its derivatives. Numerical solutions are much more readily obtained in such cases than when the functions are not separated.

An example is provided when $a = 0$ (case I), so that equation (22) then contains terms only in G and its derivatives. In all cases where the functions can be separated, the equation is thereby reduced to a Falkner-Skan type equation. The complete solutions to Falkner-Skan equations have been obtained in references 4 and 5. Thus, equation (30) (case III) and equation (32)' (case IV), in which the functions are already separated, are all Falkner-Skan equations.

Although it is not apparent from the equations in the table alone, when $a = 0$, then $u = 0$ and equation (1a) and therefore equations (21), (23), (30), and (32) disappear. In cases I and III ($W \neq 0$), when $a = 0$, the flows are straight two-dimensional flows along radial lines out from a stagnation point. In cases II and IV ($W = 0$), $a = 0$ is the trivial case of no mainstream flow.

In case III, when $a = b$, $F = G$. If, in addition, $m = 0$ and $c = -a$, equation (30) becomes equation (9.12) of reference 16:

$$F''' - F'^2 + 1 = 0 \tag{40}$$

As pointed out in reference 16, this is one of the few cases when the boundary-layer equation can be solved in closed form. The solution is

$$F' = \frac{u}{U} = 3 \tanh^2 \left(\frac{\eta}{\sqrt{2}} + \tanh^{-1} \sqrt{\frac{2}{3}}\right) - 2 \tag{41}$$

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When $c = a = b$ and $m = 0$, equation (30) becomes

$$F''' + F'^2 - 1 = 0 \tag{42}$$

Letting

$$\mathscr{F}(-\eta) = F(\eta) \tag{43}$$

then by differentiation of equation (43) and substitution, equation (42) becomes

$$\mathscr{F}''' - \mathscr{F}'^2 + 1 = 0 \tag{44}$$

and the solution thereby is seen to be

$$\mathscr{F}' = \frac{u}{U} = 3 \tanh^2 \left(\frac{-\eta}{\sqrt{2}} + \tanh^{-1} \sqrt{\frac{2}{3}}\right) - 2 \tag{45}$$

Linearity in u or w. - As discussed in reference 14 and applied in reference 13, an extension of the solutions beyond strict similarity of the velocity component can sometimes be made by addition of solutions where the boundary-layer equations are linear in u or w. Apparently such extensions are not possible for the boundary-layer flows investigated here, because equation (1) is always nonlinear in u and in w except for the typical case of no mainstream flow. Nevertheless, there is the complete freedom from specification of the form of $f(r)$ in cases III and IV. Accordingly, if

$$U_1 = a_1 f_1(r) \theta^{m-1}$$

and

$$U_2 = a_2 f_2(r) \theta^{m-1}$$

are possible solutions for mainstream flows, then

$$U = U_1 + U_2 = \left[a_1 f_1(r) + a_2 f_2(r)\right] \theta^{m-1} = af(r)\theta^{m-1}$$

likewise represents a possible mainstream flow, although solutions for u and w cannot be superimposed. This fact (the reasoning is the same for W as for U) is evident also from the ordinary differential equations for cases III (eqs. (30) and (31)) and IV (eqs. (32) and (33)), which are independent of the form of $f(r)$.

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Comparison with Experiment

An experimental investigation was made to determine whether the theory provides a reasonable approximation of the limiting flow deflections for the particular case of circular flow over a flat plate. Of the cases presented, this case is the one most likely to be encountered in actual practice.

Theoretical prediction of limiting deflection. - The case that will be investigated is case IV with $m = 1$. Under this assumption equations (32) and (33) become, respectively,

$$\frac{FF''}{2} + F''' = 0 \tag{46}$$

$$-F'G' - (F'^2 - 1) + \frac{FG''}{2} + G''' = 0 \tag{47}$$

where the following relation between the various constants has been chosen:

$$c = a = -b$$

Equation (46) is the Blasius equation, and values for F are tabulated in reference 13. Reference 12 shows that the function G' in equation (47) is expressible in the form

$$G' = P(\eta) - F'(\eta) \tag{48}$$

where $P(\eta)$ is the solution of the equation

$$P'' + \frac{FP'}{2} - F'P = -1 \tag{49}$$

with the boundary conditions

$$\lim_{\eta \to \infty} P(\eta) = 1, \; P(0) = 0$$

The solution of equation (49), however, is presented in reference 13, and the values of $P(\eta)$, ($P_1(\eta)$ in ref. 13) are presented in tables. 4234

Now the angle of flow deflection of the boundary layer at the plate surface is determined by

$$\gamma = \text{arc tan}\left(\lim_{y \to 0} \frac{w}{u}\right)$$

$$= \text{arc tan}\left(\lim_{\eta \to 0} \frac{G'}{F'}\right)$$

As $G'(0)/F'(0)$ is an indeterminate form, application of L'Hospital's rule gives

$$\gamma = \text{arc tan}\left(\lim_{\eta \to 0} \frac{G''}{F''}\right)$$

$$= \text{arc tan}\left(\lim_{\eta \to 0} \frac{P' - F''}{F''}\right)$$

$$= \text{arc tan}\left[\lim_{\eta \to 0}\left(\frac{P'}{F''} - 1\right)\right]$$

The value of $P'(0)/F''(0)$, however, is determined in reference 12 to be 4.270. Hence,

$$\gamma = \text{arc tan } 3.270 \tag{50}$$

From equation (50), the equation for the limiting deflection line on the plate surface can be found as the solution of the differential equation:

$$\frac{1}{r}\frac{dr}{d\theta} = \tan \gamma = 3.270 \tag{51}$$

Equation (51) has the solution

$$r = (\text{const.})\, e^{3.270\,\theta} \tag{52}$$

Experimental determination of limiting deflection. - The experimental determination of limiting flow deflection was made by means of smoke flow visualization (apparatus and procedures described in ref. 1) in a Lucite two-dimensional circular channel of rectangular cross section. Tests were conducted on a plate parallel to the base of the channel and fastened to the channel walls at a distance of approximately one-third the channel height. With a maximum Reynolds number of about 6×10^4, the flow for these tests was well within the laminar range.

4234 Figure 1, a view through the Lucite top of the channel, shows smoke introduced into the mainstream at approximately a midchannel position. On the test plate shown in the photograph, the circular white lines correspond to the theoretical mainstream flows. The straight lines orthogonal to the circular lines represent 10° increments in θ. The dotted lines appearing on the plate are the theoretical limiting deflection lines based on equation (52). The main-flow streamlines as depicted by the smoke trace were found to follow closely the theoretical main-flow streamlines well beyond the region of interest near the leading edge of the plate, although this is somewhat obscured by parallax in figure 1. Near the exit of the channel, some deviation of the smoke trace from the circular arcs occurred because of secondary-flow accumulations.

Figures 2(a) and (b) show limiting flow deflection determined by introducing smoke directly on the plate surface. The photographs show that the theory predicts limiting deflection very well in the range $0 \leqq \theta \leqq 10^{\circ}$. Beyond 10° the theory predicts a greater overturning of flow than that indicated by the experiment. These results give an order of magnitude to the range of values of θ where the small-angle approximation appears to be reasonable.

Prediction of limiting deflection based on translate flow. - Reference 12 presents an analysis of three-dimensional boundary-layer flows when the main-flow streamlines are translates (i.e., all streamlines are obtained from a single streamline by propagation of the streamline parallel to the plate leading edge). As a matter of interest, this theory was also used to predict limiting deflection for the present case.

In effect, a comparison can be made between the results predicted by two distinct kinds of approximations. In the small-central-angle variation method the approximation is made in the solution of the boundary-layer equations while the description of the main flow is exact. In the streamline-translate method the approximation is made in the representation of the main-flow streamlines while the solution to the boundary-layer equations obtained is exact.

For the streamline-translate method, a mean value between the radius of the inner wall of the channel and the radius corresponding to the starting point of the outer limiting line shown plotted in the photographs was chosen as the defining radius of a typical circular-arc streamline in translate flow. Since circular-arc translate flow is analyzed and the limiting flow deflection is determined in reference 12 (see fig. 5, ref. 12), the results could be applied directly in the present investigation. The basic assumption underlying such an application is that circular-arc translate flow of the type described is a reasonable approximation of the concentric-circular-arc streamline flow that actually exists in the channel.

The results of the analysis are presented in figure 2(c). A 30° section of the channel is sketched, and the theoretical limiting deflection lines for both the translate-flow analysis and the small-angle analysis are indicated. The circles appearing on the sketch represent points on the smoke flow pattern as measured from the photographs. It can be seen from figure 2(c) that the translate-flow analysis is less accurate than the small-angle analysis up to about 10°; but, from 10° on, the agreement between theory and experiment improves for the translate analysis. After 20° it appears that the predicted limiting line and the actual limiting line are very nearly parallel up to the point where wall interference causes the boundary layer to deflect in a circumferential direction.

CONCLUDING REMARKS

Exact solutions were obtained in reference 15 describing the mainstream flows over a flat or nearly flat surface for which the thin laminar-boundary-layer flows have similarity with respect to stationary polar coordinates. The solutions thus obtained were of the form

$$U = ar^n e^{m\theta}$$

$$W = br^n e^{m\theta}$$

By suitable choice of the constants a, b, n, and m, the main flows may be stagnation-point, spiral, or circular flows. The boundary layers for these exact solutions have no properly defined leading edge in the finite part of the plane, resulting in some awkwardness in relating these theoretical flows to real physical flows.

The present analysis is restricted to regions of small central angle θ. The appropriateness of the restriction in physical situations has been established experimentally (refs. 1 to 3 and 13). The mainstream flows obtained herein are

$$\left.\begin{aligned} U &= ar^n\theta^{m+1} \\ W \text{ (or } \overline{W}) &= br^n\theta^m \end{aligned}\right\} \qquad \text{(cases I and II)}$$

$$\left.\begin{aligned} U &= af(r)\theta^{m-1} \\ W \text{ (or } \overline{W}) &= bf(r)\theta^{m} \end{aligned}\right\} \qquad \text{(cases III and IV)}$$

In cases III and IV for $1 \leq m < 2$, a properly defined leading edge can be obtained along the line $\theta = 0$ where the boundary layer has zero thickness and the mainstream velocity components do not become infinite. Although different velocity distributions are obtained here, the projected main-flow streamline configurations possible are the same as those of reference 15; that is, (1) stagnation flows along radial lines from a stagnation point, (2) spiral flows out from (or in toward) a central point, or (3) circular flows.

Actual numerical solutions of the transformed boundary-layer equations are not attempted here. Particular examples are noted, however, for which solutions have been obtained elsewhere.

An experimental comparison of limiting overturning at the wall under circular mainstream flow using smoke flow-visualization techniques showed good agreement for the first 10° of the mainstream turning.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, September 26, 1956

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TABLE I. - SIMILARITY SOLUTIONS IN STATIONARY POLAR COORDINATES FOR SMALL ANGLE VARIATION

(a) Case I

U	$ar^n\theta^{m+1}$
W	$br^n\theta^m$, $b \neq 0$
η	$y\left(\frac{cr^{n-1}\theta^m}{\nu}\right)^{1/2} = y\left(\frac{cU}{\nu ar\theta}\right)^{1/2} = y\left(\frac{cW}{\nu br}\right)^{1/2}$, $c \neq 0$
Ordinary differential equations	(21) $b(n+1)(F'G'-1) - \frac{b(n+3)}{2}GF'' + a(m+1)\left[(F')^2-1\right] - \frac{a(m+2)}{2}FF'' - cF''' = 0$ (22) $am(F'G'-1) + bn\left[(G')^2-1\right] - \frac{b(n+3)}{2}GG'' - \frac{a(m+2)}{2}FG'' - cG''' = 0$
Boundary conditions	$F'(0) = G'(0) = F(0) = G(0) = 0$, $\lim_{\eta \to \infty} F'(\eta) = \lim_{\eta \to \infty} G'(\eta) = 1$
Projection of main-stream on surface	$r = C\theta^{b/a}$ spiral flow streamlines ($a \neq 0$)
Slope of projected streamline with respect to $\theta = 0$	$\dfrac{\frac{b}{a}\tan\theta + \theta}{\frac{b}{a} - \theta\tan\theta}$
Irrotationality	$m = a(n+1) = 0$
Linearity in u and w	$\left.\begin{matrix}(1a)\\(1b)\end{matrix}\right\}$ Not possible for this case
Separation of F and G	(21) Not possible for this case (22) $a = 0$
Comments and references	$a = 0$, stagnation flow: eq. (1a) and eq. (21) vanish and eq. (22) becomes a Falkner-Skan equation with solution completely known, refs. 4 and 5. Ref. 8: $a = c = n = -1$, $m = b$, $b = -c$ from ref. 8.

TABLE I. - Continued. SIMILARITY SOLUTIONS IN STATIONARY POLAR COORDINATES FOR SMALL ANGLE VARIATION

(b) Case II

U	$ar^n\theta^{m+1}$
W	0
$\overline{W}$	$br^n\theta^m$, $b \neq 0$
η	$y\left(\frac{cr^{n-1}\theta^m}{\nu}\right)^{1/2} = y\left(\frac{cU}{\nu ar\theta}\right)^{1/2}$, $c \neq 0$
Ordinary differential equations	(23) $b(n+1)\,F'G' - \frac{b(n+3)}{2}\,GF'' + a(m+1)\left[(F')^2 - 1\right] - \frac{a(m+2)}{2}\,FF'' - cF''' = 0$ (24) $am(F'G') + bn(G')^2 - \frac{b(n+3)}{2}\,GG'' - \frac{a(m+2)}{2}\,FG'' - cG''' = 0$
Boundary conditions	$F'(0) = G'(0) = F(0) = G(0) = 0,\ \lim_{\eta\to\infty} F'(\eta) = 1,\ \lim_{\eta\to\infty} G'(\eta) = 0$
Projection of main-stream on surface	$r = C$, circular-flow streamlines
Slope of projected streamline with respect to $\theta = 0^\circ$	- Cotangent θ
Irrotationality	$(n + 1) = 0$
Linearity in u and w	$\left\{\begin{matrix}1a\\1b\end{matrix}\right\}$ Not possible for this case
Separation of F and G	$\left\{\begin{matrix}23\\24\end{matrix}\right\}$ Not possible for this case
Comments and references	$a = 0$, no flow

TABLE I. - Continued. SIMILARITY SOLUTIONS IN STATIONARY POLAR COORDINATES FOR SMALL ANGLE VARIATION

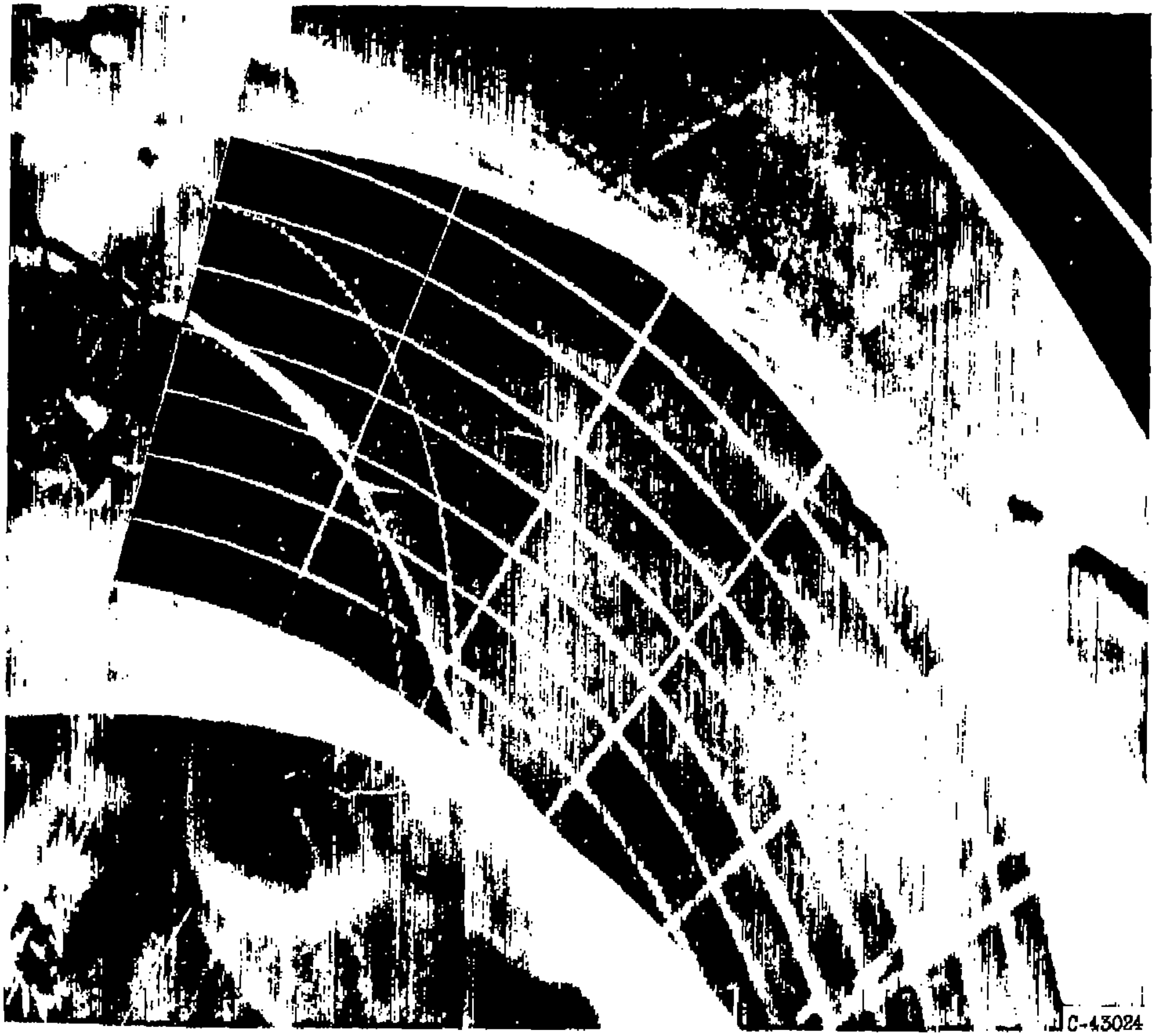
(c) Case III

U	$af(r)\theta^{m-1}$
W	$bf(r)\theta^{m}$, $b \neq 0$
η	$y\left(\frac{cf(r)\theta^{m-2}}{\nu r}\right)^{1/2} = y\left(\frac{cU}{\nu ar\theta}\right)^{1/2} = y\left(\frac{cW}{\nu br\theta^2}\right)^{1/2}$, $c \neq 0$
Ordinary differential equation	(30) $a(m - 1)\left[(F')^2 - 1\right] - \frac{am}{2} FF'' - cF''' = 0$ (31) $-\frac{a^2}{b}\left[(F')^2 - 1\right] + am(F'G' - 1) - \frac{am}{2} FG'' - cG''' = 0$
Boundary conditions	$F'(0) = G'(0) = F(0) = G(0) = 0$, $\lim_{\eta \to \infty} F'(\eta) = \lim_{\eta \to \infty} G'(\eta) = 1$
Projection of main-stream on surface	$r = Ce^{b\theta^2/2a}$, spiral flow ($a \neq 0$)
Slope of projected streamline with respect to $\theta = 0$	$\dfrac{\theta \tan \theta + \frac{a}{b}}{\theta - \frac{a}{b} \tan \theta}$
Irrotationality	$\frac{m}{r} bf(r) - af'(r) - \frac{a}{r} f(r) = 0$
Linearity in u and w	$\left\{\begin{matrix}1a\\1b\end{matrix}\right\}$ Not possible for this case
Separation of F and G	(30) Separated (31) $a = m = 0$ (see comments and references)
Comments and references	Eq. (30) is a Falkner-Skan equation, completely solved in refs. 4 and 5. If $a = 0$, eq. (1a) and eq. (24) vanish, stagnation flow, boundary conditions not achievable. If $a = b$, $F = G$: if $m = 1$, F = Blasius F Ref. 16, eq. (9:12): $a = b = -c < 0$, $m = 0$.

TABLE I. - Concluded. SIMILARITY SOLUTIONS IN STATIONARY POLAR COORDINATES FOR SMALL ANGLE VARIATION

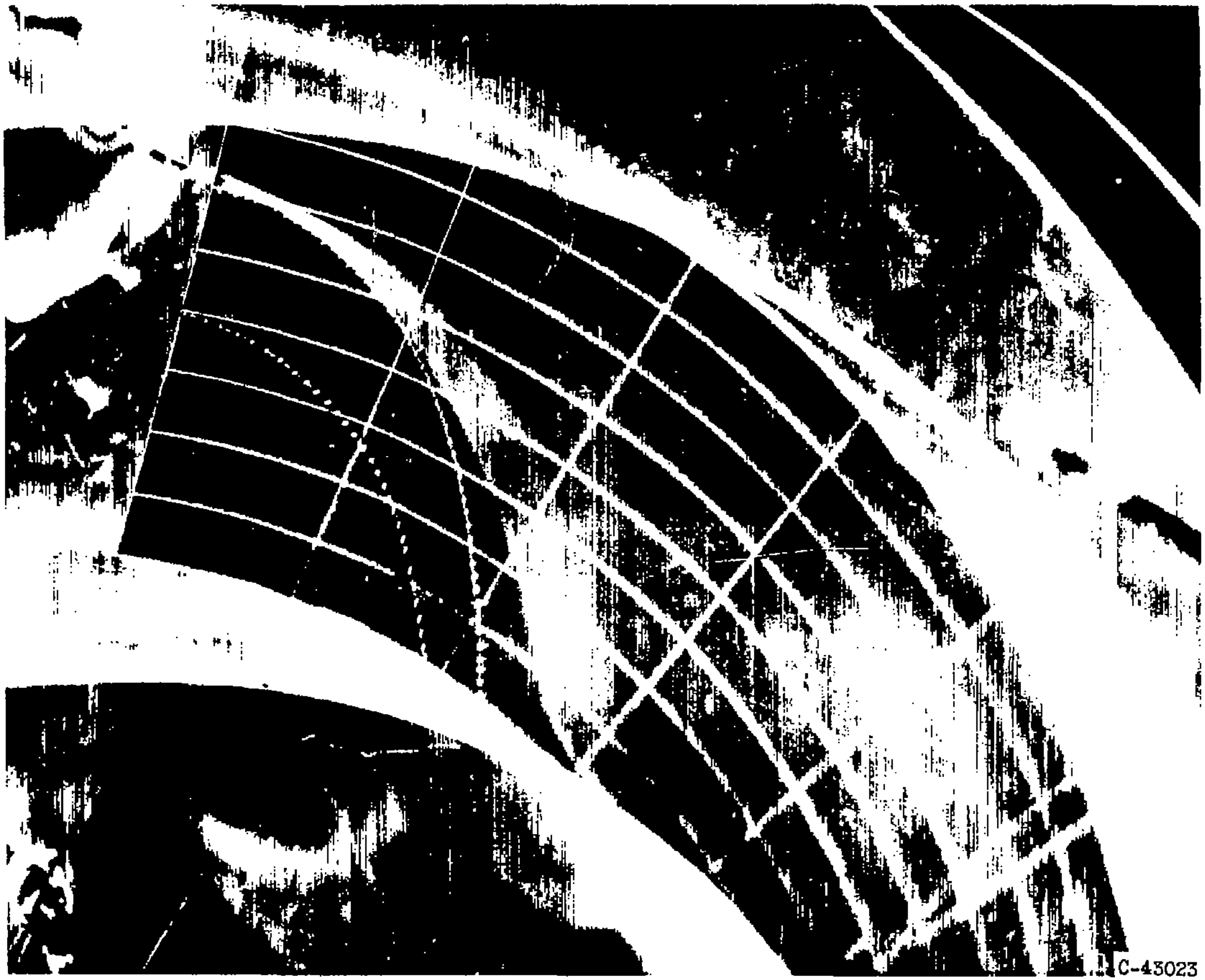
(d) Case IV

U	$af(r)\theta^{m-1}$
W	0
$\overline{W}$	$bf(r)\theta^{m}$, $b \neq 0$
η	$y\left(\frac{cf(r)\theta^{m-2}}{\nu r}\right)^{1/2} = y\left(\frac{cU}{\nu ar\theta}\right)^{1/2}$, $c \neq 0$
Ordinary differential equation	(32) $a(m - 1)\left[(F')^2 - 1\right] - \frac{am}{2} FF'' - cF''' = 0$ (33) $-\frac{a^2}{b}\left[(F')^2 - 1\right] + am(F'G') - \frac{am}{2} FG'' - cG''' = 0$
Boundary conditions	$F'(0) = G'(0) = F(0) = G(0) = 0$, $\lim_{\eta\to\infty} F'(\eta) = 1$, $\lim_{\eta\to\infty} G'(\eta) = 0$
Projection of main-stream on surface	$r = C$, circular-flow streamlines
Slope of projected streamline with respect to $\theta = 0$	- Cotangent θ
Irrotationality	$f'(r) + \frac{1}{r} f(r) = 0$
Linearity in u and w	$\left.\begin{matrix}(1a)\\(1b)\end{matrix}\right\}$ Not possible for this case
Separation of F and G	(32) Separated (33) $a = 0$ (See comments and references)
Comments and references	Eq. (32) is a Falkner-Skan equation completely solved in refs. 4 and 5. If $a = b$, $F \neq G$ (see boundary conditions) If $m = 1$, F = Blasius F Ref. 9: $c = a = b$, $m = 1$. If $a = 0$, no flow.



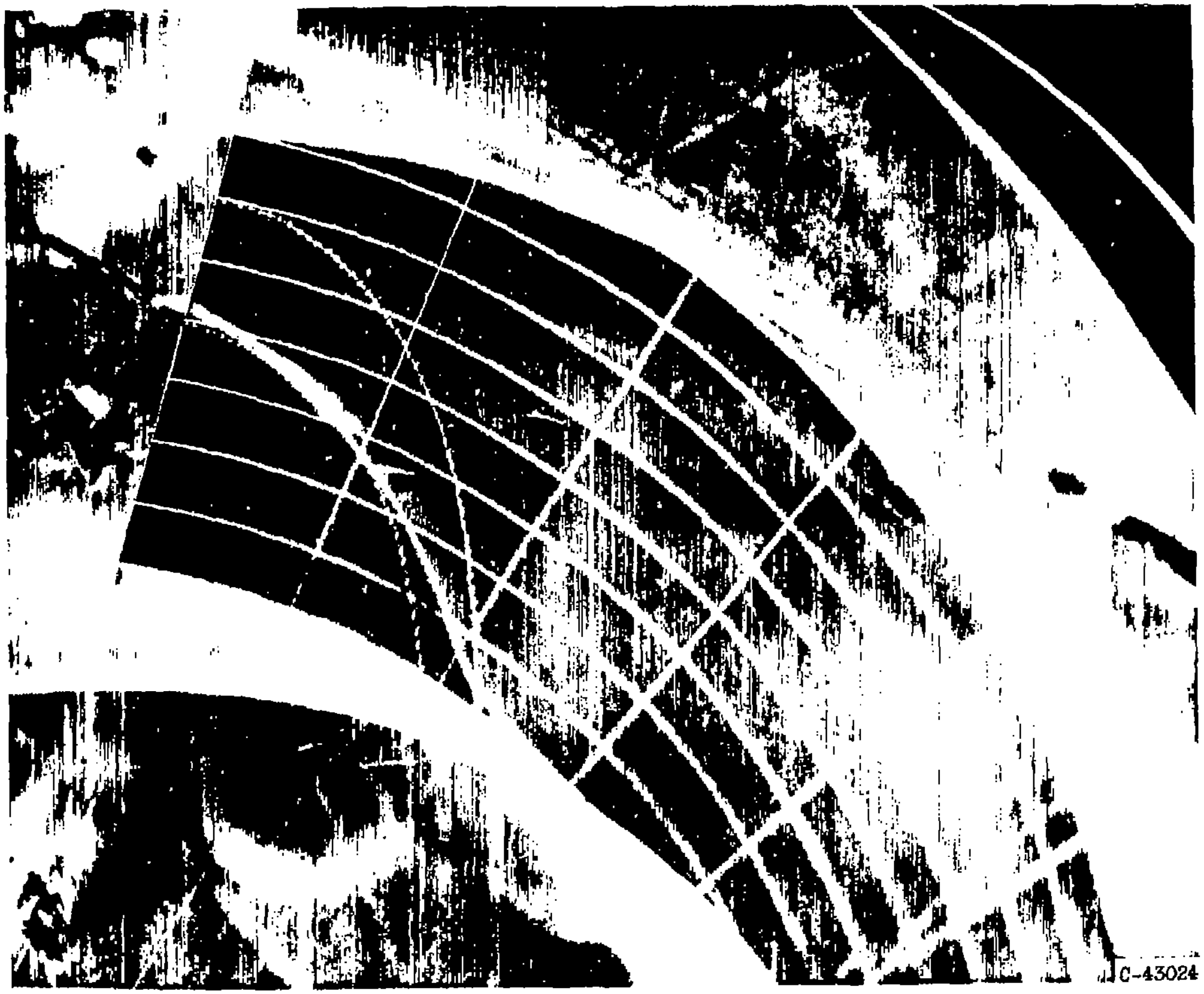
(b) Boundary-layer streamline near midchannel at inlet.

Figure 2. - Continued. Limiting flow deflection in circular channel.



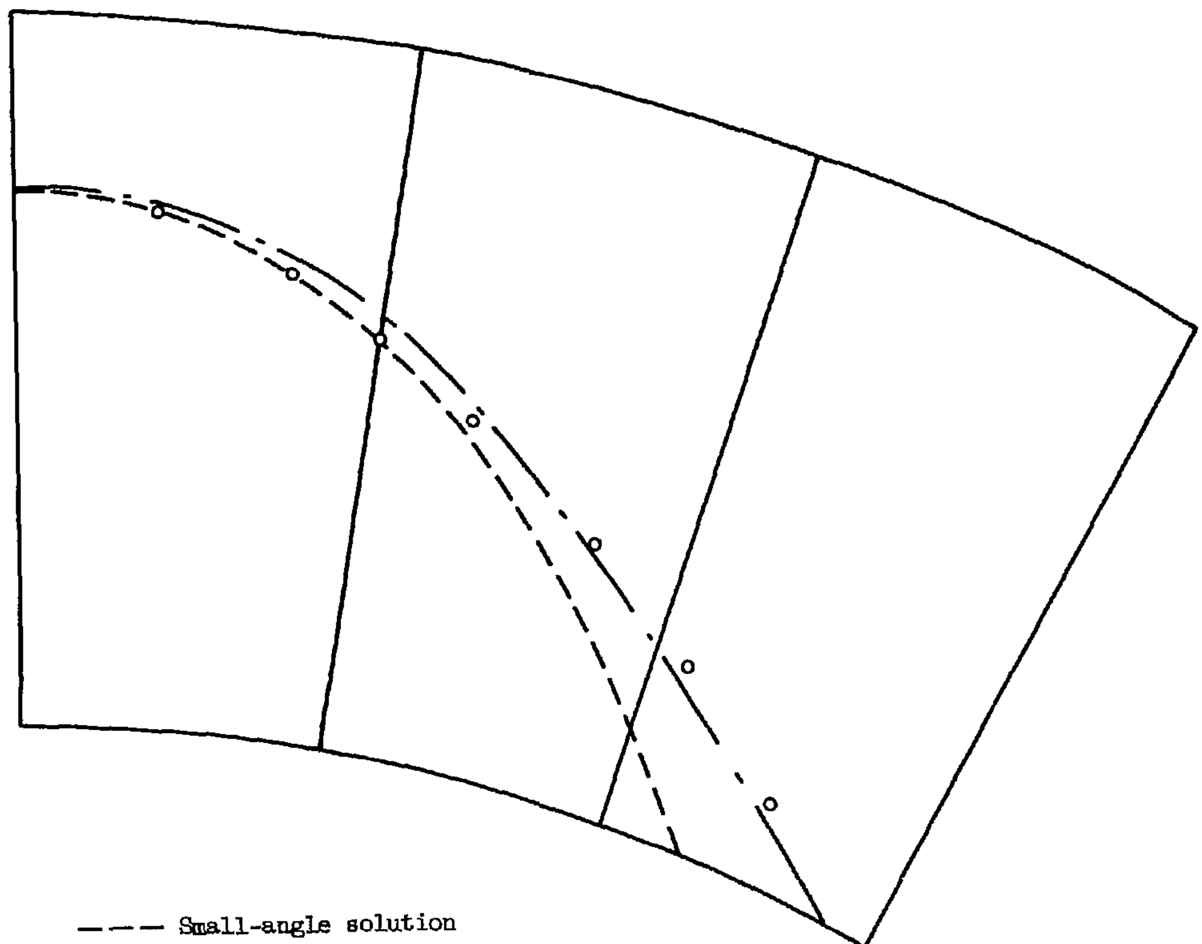
(a) Boundary-layer streamline near pressure surface at inlet.

Figure 2. - Limiting flow deflection in circular channel.



(b) Boundary-layer streamline near midchannel at inlet.

Figure 2. - Continued. Limiting flow deflection in circular channel.



— — — Small-angle solution
—·— Translate solution
o Points on smoke flow pattern

(c) Comparison of translate solution with small-angle solution and smoke pattern.

Figure 2. - Concluded. Limiting flow deflection in circular channel.